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Synthesis of fractal signals with wavelet bases*

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Abstract Based on the characterization of self-similarity, a new model of the fractal signals with discrete wavelet synthetic formula is obtained, and a formula to calculate the time-average spectrum of the signals constructed by the new model is given. It is shown that the new model is more precise than the previous ones.

Keywords: **self-similarity, wavelet, fractal signal, spectrum.**

In this paper, we describe how to use the synthesis formula of discrete wavelet transform to represent a fractal signal. In contrast with traditional methods[1~4], wavelet transforms generally make the analysis easier. So the wavelet analysis of $1/f$-type signals is very important in many applications. Recently, Wornell[5] has studied the relationship between orthonormal wavelet basis and nearly -$1/f$ models, and presented a representation of $1/f$-type signals. He shows that wavelet expansion in terms of uncorrelated random variables can constitute models for $1/f$-type signals. His method is simple but disregards the dependencies among wavelet coefficients. In fact, such dependencies have effect on the spectrum of the constructed signals (refer to the experiments in Sec. 3). We develop a new method for $1/f$-type signals, in which the generation of $1/f$-type signals relies on the correlation structure of wavelet coefficients and the spectral parameter γ. A comparison between Wornell's method[5] and ours shows that the latter is more precise in approaching the $1/f$-type signals.

1 Wavelet transform of fractal signal

Let $f(t)$ be a random process, and $-\infty < t < \infty$. For any positive real number a, if the equalities

$$E[f(at)] = a^{H}E[f(t)] \quad \text{and} \quad E[f(at)f(as)] = a^{2H}E[f(t)f(s)] \tag{1}$$

are satisfied, then $f(t)$ is a self-similar process (signal) with the parameter H, where $E[\cdot]$ denotes the mathematical expectation of a function. If $f(t)$ is a self-similar process, and has power spectrum obeying the following relationship:

$$S_f(\omega) \propto \frac{1}{|\omega|^{\gamma}}, \tag{2}$$

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then $f(t)$ is called a $1/f$ process (or a fractal signal), where $\gamma = 2H + 1$ is a spectra parameter. The self-similarity is one of the important properties of a $1/f$ process. It means that the statistics of a $1/f$ process do not change when we stretch or shrink the time axis. That is, the process lacks a characteristic scale: the behavior of the process on short time scales is the same as its behavior on the long ones. It is noted that for a signal's discrete wavelet transform, the scales take discrete values 2^{-m}, $m \in \mathbb{Z}$. So we need to make a further explanation for the above definition. In the following discussions, the self-similarity of the signal $f(t)$ means that Relation (1) is satisfied for all the discrete real number $a \in \{2^{-m} \mid m \in \mathbb{Z}\}$.

Let $\psi(t)$ be a basic wavelet and $\overline{\psi(t)}$ its conjugate basic wavelet. Then the discrete wavelet transform of a signal $f(t)$ can be defined by[6]

$$x(m,n) = \int_{-\infty}^{\infty} f(t)\,\overline{\psi_{m,n}}(t)\mathrm{d}t, \qquad m,n \in \mathbb{Z}, \tag{3}$$

where $\psi_{m,n}(t) = 2^{m/2}\psi(2^m t - n)$. If $f(t)$ satisfies the admissibility condition, the representation of $f(t)$ is

$$f(t) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} c_{m,n}\psi_{m,n}(t), \tag{4}$$

where $c_{m,n} = \langle f(t), \tilde{\psi}_{m,n}(t)\rangle$, and $\tilde{\psi}(t)$ is a dual wavelet of $\psi(t)$. The wavelet transform between two scales also possesses self-similarity. For this reason, the fractal signals treated by wavelet transform have aroused a great deal of attention[7~9]. In this study we will model $1/f$-type signals using Eq. (4). The characterization of self-similarity for $1/f$ signals by wavelet transform is to be discussed first.

If $\psi(t)$ is an orthonormal wavelet and the wavelet transform of a signal $f(t)$ is defined by Eq. (3), $x(m,n)$ are just its wavelet coefficients, i.e. $c_{m,n} = x(m,n)$. By Eq. (1), it is easy to prove that $f(t)$ is a self-similar signal with parameter H, if and only if for any $k, m, n \in \mathbb{Z}$ and $\forall\, m_i, n_i \in \mathbb{Z}, i = 1,2$ hold such that

$$E[x(m-k,n)] = 2^{(2H+1)k/2}E[x(m,n)], \tag{5}$$

and

$$E[x(m_1-k,n_1)x(m_2-k,n_2)] = 2^{(2H+1)k}E[x(m_1,n_1)x(m_2,n_2)]. \tag{6}$$

Equation (6) establishes the relationship among the wavelet coefficients $x(m,n)$ between two scales. If we compare the wavelet coefficients at any scales with the wavelet coefficients at scale 1 (corresponding to $m=0$), this relationship is more obvious. Let $R_{m1,m2}(n_1,n_2) = E[x(m_1,n_1)x(m_2,n_2)]$, and $R_m(\cdot)$ denote $R_{m1,m2}(\cdot)$ when $m = m_1 = m_2$. Then $R_m(\cdot)$ and its normalization $r_m(\cdot)$ can be written as

$$R_m(n_1,n_2) = 2^{-m(2H+1)}R_0(n_1,n_2), \tag{7}$$

$$r_m(n_1,n_2) = R_m(n_1,n_2)/\sqrt{\mathrm{var}(x(m,n_1))\mathrm{var}(x(m,n_2))}$$

$$= R_0(n_1, n_2)/\sqrt{\mathrm{var}(x(0, n_1))\mathrm{var}(x(0, n_2))} = r_0(n_1, n_2). \tag{8}$$

The above expressions verify that the correlation structure of the wavelet coefficients for self-similar signals is independent of scales, i.e. the property of the second-order statistic is invariable with scale.

2 Modeling for fractal signals

Let $\xi(m, n)$ be a zero mean 2-D random process. Define

$$f(t) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} \xi(m, n)\psi_{m,n}(t) \tag{9}$$

as a model for modeling $1/f$ signals. By the discussion in Sec. 1, the process $\xi(m, n)$ ought to satisfy Relationships (5) and (6), and to be wide-sense stationary in n if we expect that $f(t)$ is of self-similarity. To calculate the spectrum of $f(t)$, we consider a resolution-limited approximation to $f(t)$

$$f_M(t) = \sum_{m=-M}^{M}\sum_{n=-\infty}^{\infty} \xi(m, n)\psi_{m,n}(t),$$

that is, the information at resolutions coarser than 2^M and finer than 2^{-M} is discarded, where M is a sufficiently large positive integer. By Eqs. (5) and (6), the power of the signal $f(t)$ at each scale is infinite. So we have to consider the time-averaged correlation functions as follows:

$$R_{f_M}(\tau) = 2^{-M}\int_0^{2^M} E[f_M(t)f_M(t+\tau)]\mathrm{d}t.$$

Using R_{f_M} and $f(t) = \lim_{M\to\infty} f_M(t)$, we have already proved the following theorem.

Theorem 1. Let $\psi(t)$ be an orthonormal wavelet with the Rth-order regularity, $R > H + 1/2 > 0$, and $\xi(m, n)$ satisfies Eqs. (5) and (6) as well as the following conditions: (i) for any $m \in \mathbb{Z}$, $\xi(m, n)$ is wide-sense stationary in n; (ii) $\sum_{k=0}^{\infty} |R_0(k)| < \infty$, where $R_0(k) = E[\xi(0, n+k)\xi(0, n)]$; (iii) for any $m_1, m_2 \in \mathbb{Z}$, $E[\xi(m_1, n_1)\xi(m_2, n_2)] = 0$ if $m_1 \neq m_2$. Then $f(t)$ constructed by Eq. (9) is a self-similar signal with parameter H and has a time-averaged spectrum

$$S_f(\omega) = \sum_{m=-\infty}^{\infty} 2^{-m\gamma} S_0(2^{-m}\omega) |\hat{\psi}(2^{-m}\omega)|^2, \tag{10}$$

where $S_0(\omega) = \sum_{k=-\infty}^{\infty} R_0(k)\mathrm{e}^{-\mathrm{j}k\omega}$, $\gamma = 2H + 1$, and $\hat{\psi}(\omega)$ denotes the Fourier transformation of $\psi(t)$.

Since $R_m(k) = 2^{-m\gamma}R_0(k)$, we have $S_m(\omega) = 2^{-m\gamma}S_0(\omega)$. Hence the general terms in Eq. (10) may be considered as a spectrum of $f(t)$ at corresponding scales, and $S_f(\omega)$ as the superposition of an infinite number of such spectra. Note that the spectrum $S_f(\omega)$ of the signal constructed by Eq. (9) relates to not only the spectral parameter γ but also the correlation structure $R_m(k)$ among $\xi(m, n)$ at each scale, which differs from the previous model[5]. On the other hand, $S_f(\omega)$ has octave-spaced ripple which is uniform-spaced on a log-log frequency plot, i.e. for any $m \in \mathbb{Z}$,

$$S_f(2^m\omega) = 2^{-m\gamma}S_f(\omega).$$

This ripple essentially results from the discretization for the scales[10]. So it is not avoidable when using Eq. (9) to generate $1/f$-type signals. For this reason, the generated signal is always a nearly-$1/f$ signal. However, Eq. (9) with a orthonormal wavelet basis constitutes a model better than the others.

If $S_0(\omega) > 0$ for $|\omega| > 0$, we can easily show that the spectrum of $S_f(\omega)$ is bounded in the sense that

$$m_f/|\omega|^\gamma \leqslant S_f(\omega) \leqslant M_f/|\omega|^\gamma \tag{11}$$

for some $0 < m_f \leqslant M_f < \infty$, i.e. the generated signal $f(t)$ is a nearly-$1/f$ signal.

3 Experimental results

It is clear that the positive numbers M_f and m_f in Eq. (11) characterize the error between the spectrum $S_f(\omega)$ and the desired $1/f$ spectrum. So the error of the generated signal $f(t)$ is defined as $E = M_f - m_f$, where $m_f = \inf(|\omega|^\gamma S_f(\omega))$ and $M_f = \sup(|\omega|^\gamma S_f(\omega))$.

Let $\xi(m,n)$ be a collection of mutually uncorrelated, zero-mean random variables with variances $\mathrm{var}(\xi(m,n)) = \sigma^2 2^{-\gamma m}$. Then the signal generated by Eq. (9) is nearly-$1/f$[5]. In our method, we first need to generate a process $\xi(m,n)$. Generally, $\xi(m,n)$ may have a complicated correlation structure like Eq. (6). We can, however, choose a simple modeling for such a correlation structure. Here for each m, $\xi(m,n)$ is chosen to be a Markov process that is both simple and effective in many applications. Such a process, for fixed m, satisfies the following first-order autoregressive model:

$$x_1 = w_1,\ x_n - \rho x_{n-1} = w_n, \quad n > 1,\ 0 < \rho < 1, \tag{12}$$

where w_n is a zero-mean and stationary white Gaussian noise with covariance $\mathrm{var}(w_n) = (1-\rho^2)2^{-\gamma m}$. In this case, $\xi(m,n)$ is the output of white Gaussian noise through a linear time-invariant system. For fixed m, it has correlation function with the form $R_m(k) = 2^{-\gamma m}e^{a|k|}$, where a is related to the parameter ρ, and $a > 1$. It is easily shown that $\xi(m,n)$ satisfies the conditions of Theorem 1. However, the parameter a (or ρ) will have effect on the spectrum $S_f(\omega)$. So we need to choose a suitable parameter a for different γ. Table 1 gives the experimental results with different γ when $\psi(t)$ is the Haar wavelet or the second-order Daubechies wavelet, where E_{wor} is the error of the signals generated by Wornell's method and E_{our} is the error of the signals generated by our method.

It is evident in table 1 that with the different parameters γ the errors E_{our} of the signals constructed by our method are always less than E_{wor} by the Wornell's method. In addition, as the spectral parameter γ increases, the parameter a decreases, showing that the correlation of $\xi(m,n)$ increases with the increase in γ.

Table 1 Comparison between two methods

Error		Haar wavelet				Second-order Daubechies wavelet		
	γ	0.5	1.0	1.5	1.8	1.5	1.8	2.0
	a	2.0	1.8	1.5	1.7	2.8	2.4	2.2
E_{wor}		0.45	1.77	7.01	17.14	1.75	4.52	8.61
E_{our}		0.12	0.50	3.33	12.24	0.80	2.03	4.48

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